Quantum-classical correspondence in perturbed chaotic systems

Giuliano Benenti[1] and Giulio Casati[1,2]

[1]*International Center for the Study of Dynamical Systems, Università degli Studi dell'Insubria
and Istituto Nazionale per la Fisica della Materia, Unità di Como, Via Valleggio 11, 22100 Como, Italy*
[2]*Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy*

We discuss the behavior of fidelity for a classically chaotic quantum system. We show the existence of a critical value of the perturbation above which the quantum decay, exponential or power law, follows the classical one. The independence of the decay rate of the perturbation strength, discussed in the literature, is a consequence of the quantum-classical correspondence of the relaxation process.

Quantum chaos, namely, the attempt to understand classical dynamical chaos in terms of quantum mechanics, has led to a much better understanding of some properties of quantum motion which go beyond simple integrable models and perturbative treatments. A simple property of quantum conservative Hamiltonian systems with a finite number of particles, namely, the discrete spectrum, has been at the origin of some difficulties. Indeed, in the classical ergodic theory a discrete spectrum together with linear local instability of motion is a typical feature of integrable systems, while chaotic systems are characterized by a continuous spectrum and exponential local instability. This fact has cast doubt on the possibility of dynamical chaos in quantum mechanics. On the other hand, the correspondence principle requires transition to classical mechanics of all properties, including dynamical chaos. As discussed on several occasions [1] this apparent contradiction is resolved by taking into account that a sharp distinction between the discrete and continuous spectrum becomes meaningful only in the limit $t \to \infty$. For finite times, there exist different time scales below which the quantum motion can display chaotic properties like the corresponding classical one. These time scales tend to infinity as the effective Planck constant $\hbar_{\mathrm{eff}} \to 0$. Two time scales are of particular importance: the random or Ehrenfest time scale t_r and the relaxation or Heisenberg time scale t_R. For $t < t_r$ the quantum motion is exponentially unstable like the classical one, while the quantum relaxation process takes place during the time $t < t_R$. Since typically $t_r \ll t_R$, the quantum relaxation process takes place in the absence of exponential instability. A clear illustration of this peculiar feature of quantum motion is shown in [2]. It should be remarked that this lack of exponential instability does not prevent exponential decay of dynamical quantities like correlation functions or survival probability [3].

Recently the problem of the stability of quantum motion has attracted much interest, also in relation to the field of quantum computation. A quantity of central importance which has been the focus of many studies [4–12] is the so-called fidelity $f(t)$, which measures the accuracy to which a quantum state can be recovered by inverting, at time t, the dynamics with a perturbed Hamiltonian:

$$f(t) = |\langle \psi | e^{i\hat{H}t} e^{-i\hat{H}_0 t} | \psi \rangle|^2. \tag{1}$$

Here ψ is the initial state which evolves for a time t with the Hamiltonian $\hat{H}_0$ while $\hat{H} = \hat{H}_0 + \hat{V}$ is the perturbed Hamiltonian. The analysis of this quantity has shown that, under some restrictions, the decay of $f(t)$ is exponential with a rate given by the classical Lyapunov exponent [5]. This result appears to be consistent with recent experiments on the polarization echoes in nuclear magnetic resonance [13] and with numerical computations [6]. More recent papers have contributed to clarifying different complementary aspects of the problem [7,9–11], including the relation to the local density of states [12] and the use of the semiclassical approach [8]. The analysis of this quantity has some delicate aspects concerning attempts to characterize quantum chaos via the classical Lyapunov exponent and the role of the above mentioned time scales. It is therefore highly desirable to have very accurate numerical results, and to this end it is necessary to consider simple systems which display the generic features of classical and quantum chaotic systems and which can be easily treated numerically.

In this paper we consider the behavior of fidelity for a classically chaotic system, in the delocalized regime of quantum ergodicity, in which the wave functions have a complex pattern that can be described within the framework of random matrix theory. We show that the type of decay and its rate depend on the strength of the perturbation. In particular, above a critical border, the quantum decay mimics, up to the relaxation time scale, the classical one, which, in turn, can be exponential or power law. The independence of the decay rate of the perturbation, discussed in several papers, simply reflects the properties of the underlying classical motion.

We consider the classical sawtooth map

$$\bar{n} = n + k_0(\theta - \pi), \quad \bar{\theta} = \theta + T\bar{n}, \tag{2}$$

where (n, θ) are conjugated action-angle variables ($0 \le \theta < 2\pi$), and the overbars denote the variables after one map iteration. Introducing the rescaled momentum variable $p = Tn$, one can see that the classical dynamics depends only on the single parameter $K_0 = k_0 T$. We consider the map (2) on the torus $-\pi L \le p < \pi L$, where L is an integer. For $K_0 > 0$ the motion is completely chaotic and diffusive, with Lyapunov exponent given by $\lambda = \ln\{(2 + K_0 + [(2 + K_0)^2$

$-4)^{1/2}]/2\}$. For $K_0 > 1$ the diffusion coefficient is well approximated by the random phase approximation, $D \approx (\pi^2/3)K_0^2$.

The quantum evolution in one map iteration is described by a unitary operator $\hat{U}_0$ acting on the wave function ψ:

$$\bar{\psi} = \hat{U}_0 \psi = e^{-iT\hat{n}^2/2}e^{ik_0(\hat{\theta}-\pi)^2/2}\psi, \qquad (3)$$

where $\hat{n} = -i\partial/\partial\theta$ (we set $\hbar = 1$). We take $-N/2 \leq n < N/2$, $k_0 = (K_0/2\pi L)N$, $T = 2\pi L/N$. The classical limit corresponds to $N \to \infty$. We note that in this simple quantum model one can observe important physical phenomena like dynamical localization and cantori localization [14]. Our aim is to study the fidelity decay in the delocalized regime of quantum ergodicity. Moreover, we start by considering parameter values for which there is no initial transient diffusive behavior, which may considerably affect the decay of fidelity.

In order to compute the fidelity we choose to perturb our system by slightly varying the kicking strength $K = K_0 + \epsilon$, with $\epsilon \ll K_0$. Correspondingly, the perturbed quantum kicking parameter is $k = k_0 + \sigma$, with $\sigma = \epsilon N/(2\pi L)$. Since we want to compare classical and quantum evolution, we compute the classical "fidelity" $f_c(t)$ in the following way: we consider in the phase space a uniform density of points inside a strip of area $A = 2\pi\nu$ ($0 \leq \theta < 2\pi$, $-\nu/2 \leq p < \nu/2$). We then define $f_c(t)$ as the overlap of the initial area A with the area A' obtained by evolving A for t iterations of the map (2) and then reversing the evolution for t iterations with the perturbed strength $K = K_0 + \epsilon$. In practice, we follow the evolution of 10^6 trajectories uniformly and randomly distributed inside the area A and define the fidelity $f_c(t)$ as the percentage of orbits that return back to the area A at time t, after the above reversing procedure. The corresponding quantum initial condition is given by a uniform mixture of momentum states located inside the area A. We note that this choice, in addition to giving the correct classical limit when $N \to \infty$, introduces a convenient averaging procedure. Moreover, we have checked that the same fidelity decay rates are obtained if one starts from pure states, like momentum eigenstates or coherent states.

The behavior of the classical fidelity is shown in Fig. 1, for $K_0 = 1$, $L = 1$, and different values of the perturbation strength ϵ. In this particular regime, characterized by (i) uniform local exponential instability and (ii) absence of diffusive regime, the fidelity decay is ruled by the Lyapunov exponent λ. The exponential decay starts after an initial transient time $t \approx \ln(\nu/\epsilon)/\lambda$, which is required to amplify the perturbation up to the scale ν [15].

The decay of the quantum fidelity is Gaussian below a perturbative border [7,8]. This border is given by the value of the perturbation at which the typical transition matrix element U between quasienergy eigenstates becomes larger than the average level spacing $1/\rho$. For ergodic eigenfunctions, $U \sim \sigma/\sqrt{N}$, while the density of quasienergy states is given by $\rho = N/2\pi$. Therefore the perturbative border is given by $\sigma_p \approx 1/\sqrt{N}$. Above this border one typically expects an exponential decay of fidelity, with a rate $\Gamma = 2\pi\rho U^2 \approx \sigma^2$ given by the width of the Breit-Wigner local density of states [7].
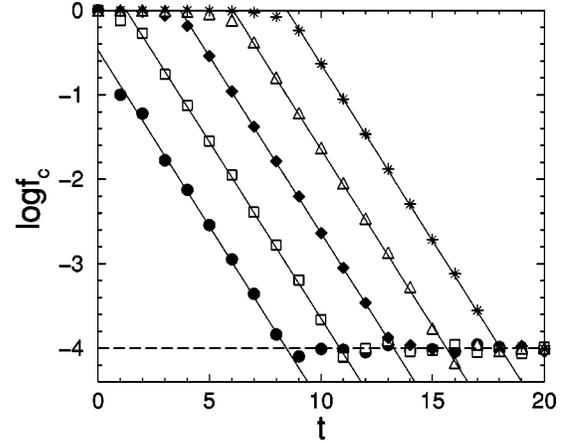
FIG. 1. Decay of classical fidelity for the classical sawtooth map with $K_0 = 1$, $L = 1$, $\nu = 2\pi/10^4$, and perturbation strength $\epsilon = 10^{-3}$ (circles), 10^{-4} (squares), 10^{-5} (diamonds), 10^{-6} (triangles), and 10^{-7} (stars). The straight lines show the decay $f_c(t) \propto \exp(-\lambda t)$, with Lyapunov exponent $\lambda = 0.96$. The dashed line indicates the saturation value $f_{c,\infty} = \nu/(2\pi L) = 10^{-4}$. Here and in the following figures the logarithms are decimal.

This theoretical prediction is confirmed in Fig. 2, which shows the decay of quantum fidelity at $\epsilon = 5 \times 10^{-5}$ and different N values, with $\sigma > \sigma_p$. The nice scaling behavior of Fig. 2 confirms the predicted exponential decay $f(t) \approx \exp(-C\sigma^2 t)$, with the numerically determined constant $C \approx 2.2$.

On the other hand, as stated in the Introduction, one expects that in the semiclassical regime the quantum motion mimics the classical one up to the relaxation time scale which is determined by the density of quasienergy eigenstates that significantly contribute to the wave function dynamics. To this end it is necessary that the perturbation σ is strong enough to allow the quantum motion to follow, on the
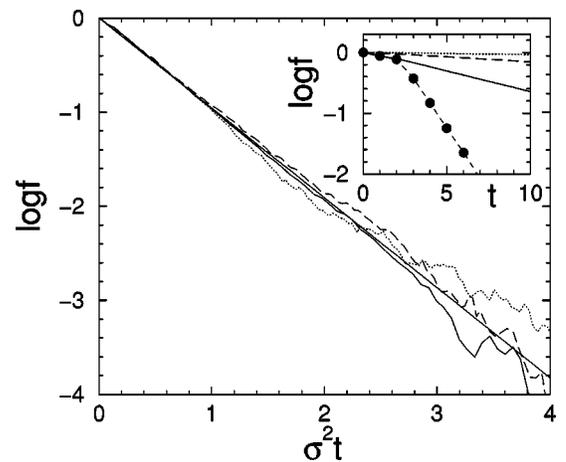
FIG. 2. Decay of the fidelity for the quantum sawtooth map at $K_0 = 1$, $L = 1$, $\epsilon = 5 \times 10^{-5}$, $N = 8192$ (dotted line, $\sigma = 0.065$), 16 384 (dashed line, $\sigma = 0.13$), and 32 768 (solid line, $\sigma = 0.26$). The straight line gives the decay $f(t) = \exp(-\Gamma t)$, with rate $\Gamma = 2.2\sigma^2$. As initial state we take a momentum eigenfunction with $n = 0$. The inset shows a magnification for small times. Circles give the classical decay for $\nu = 10^{-4}$.
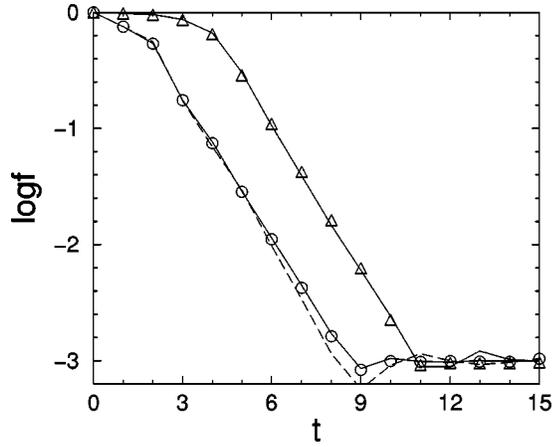
FIG. 3. Classical and quantum fidelity decay for $K_0 = 1$, $L = 1$, $\nu = 2\pi/10^3$. Left curves: $\epsilon = 10^{-3}$, $N = 16\,384$ (dashed line, $\sigma = 2.61$), $N = 131\,072$ (solid line, $\sigma = 20.9$), and classical decay (circles). Right curve: $\epsilon = 10^{-4}$, $N = 131\,072$ (solid line, $\sigma = 2.09$) and classical decay (triangles).

FIG. 4. Rate γ of the exponential decay for the quantum fidelity versus perturbation strength σ, for $K_0 = 1$, $N = 2048$ (circles), 8192 (diamonds), and 65 536 (squares), $K_0 = 2$, $N = 8192$ (stars), $K_0 = 10$, $N = 8192$ (triangles). The dashed line gives the decay rate $\Gamma = 2.2\sigma^2$. The solid lines show the Lyapunov decay, with rates $\lambda = 0.96$ (at $K_0 = 1$), 1.32 ($K_0 = 2$), and 2.48 ($K_0 = 10$).

average, the initial classical decay. In our case this may happen if σ is large enough to induce transitions at least between nearest neighbor momentum states, namely, $\sigma > \sigma_c \approx 1$. If $\sigma < \sigma_c$, the quantum excitation is unable to follow the classical spreading of the initial state. One may also argue in a different way: since with our choice of parameters we are in the metallic regime, all N quasienergy states are involved in the evolution of the unperturbed system. Then the effect of the perturbation on the quantum motion can imitate the corresponding classical one only if there are no quantum localization effects on the quasienergy states. This happens when the width of the local density of states becomes comparable to the bandwidth, that is, $\rho\Gamma \approx N$, which again gives the threshold value $\sigma_c \approx 1$. We remark that, as discussed in [16], in the theory of Wigner band random matrices the Breit-Wigner regime corresponds to a sort of partial perturbative localization. The above theoretical estimate is well confirmed by our numerical data presented in Figs. 3 and 4. Figure 3 shows that for $\sigma > 1$ the quantum fidelity closely follows the classical behavior, namely, it decays exponentially with the classical rate given by the Lyapunov exponent. Figure 4 shows the decay rate γ as a function of the perturbation strength σ. It is clearly seen that for $\sigma < 1$ the decay rate is proportional to σ^2, that is, to the width of the Breit-Wigner. Therefore $\sigma_c \approx 1$ is a critical value which separates two distinct regimes: a pure quantum perturbation dependent regime, and a semiclassical regime. We note that the perturbation σ depends on the product $N\epsilon$. For $\sigma > 1$ the decay rate does not change by increasing N at fixed ϵ, since by doing this we go toward the classical limit. On the other hand, if we increase ϵ at fixed N (provided that the perturbation remains classically small, i.e., $\epsilon \ll K_0$) the decay rate also does not change, since the exponential amplification of the perturbation is controlled by the parameter $K \approx K_0$. In both cases the *decay rate* of fidelity is perturbation independent. This is a property of the classical motion which, in the semiclassical regime, is shared by quantum mechanics. However, we would like to stress that the overall decay remains *perturba-*

tion dependent, since the exponential decay starts after a time $\propto |\ln\epsilon|$ (see Figs. 1 and 3).

For the parameter values of Figs. 1–4, the decay of fidelity is exponentially fast and the saturation value $f_\infty = \nu/(2\pi L)$ is reached on times of the order of the Ehrenfest time $t_r \approx \ln N/\lambda$, which is much shorter than the Heisenberg time. In order to observe the effect of the Heisenberg time scale it is necessary to have a much slower decay of fidelity. In Fig. 5 we take $K_0 = 1$ and $L = 50$, so that we allow for a Gaussian diffusive process in momentum space. Because of this, during the diffusion time the fidelity decays in the classical case as $1/\sqrt{Dt}$ [17]. Figure 5 shows that for $\sigma > \sigma_c \approx 1$ the quantum decay follows the classical one for larger and larger times as N increases, in agreement with the corre-
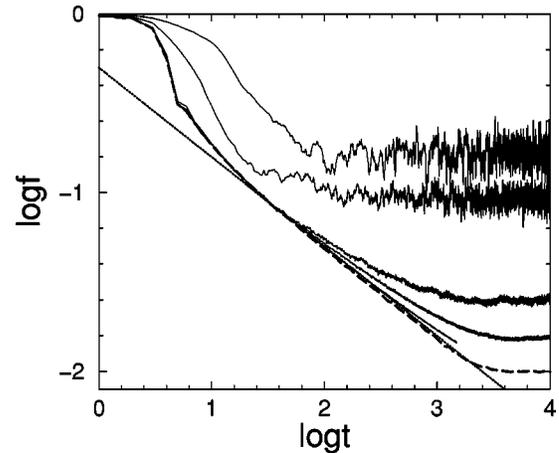
FIG. 5. Fidelity decay in the diffusive regime with $L = 50$, $K_0 = 1$, $\nu = \pi$, $\epsilon = 0.05$. The dashed curve gives the classical decay, the solid curves the quantum decay: from top to bottom $N = 1024$ ($\sigma = 0.16$), 2048 ($\sigma = 0.33$), 8192 ($\sigma = 1.30$), 16 384 ($\sigma = 2.61$), and 32 768 ($\sigma = 5.22$). The straight line indicates the decay $f(t) \propto 1/\sqrt{Dt}$.

spondence principle. The asymptotic value is $f_\infty = \nu l/(2\pi L)$, where, according to the scaling theory of localization, $l = \xi/N = g(x)$, with $x = k^2/N$ [18]. Here ξ is the actual localization length of the "sample" of size N, while k^2 gives the localization length for the infinite sample, up to a numerical constant of order 1. The scaling function $g(x)$ is proportional to x for $x \ll 1$ and saturates to 1 for $x \gg 1$. The transition value $x = 1$ corresponds to $N \approx 10^5$. Moreover, the saturation value is approached after a relaxation time $t_l \approx \xi$. We stress that in the case of Fig. 5 the decay of fidelity is controlled by the diffusion coefficient and not by the Lyapunov exponent. The observation of this regime represents a challenge for experiments like spin echoes. Further theoretical investigations are also desirable in order to understand more clearly the effect of classical diffusion and quantum localization on the behavior of fidelity.

In summary, we have shown that the decaying behavior of fidelity in a classically chaotic system strongly depends on system parameters as well as on the perturbation strength.

Nevertheless, there is a regime in which the decay *rate* (exponential or power law) is perturbation independent: in this regime the quantum motion simply mimics the properties of the underlying classical dynamics. Therefore the decay of fidelity cannot be taken as a characterization of quantum chaos, as was sometimes proposed [4,6]. We emphasize that the quantum to classical correspondence of the average behavior is valid until the Heisenberg time scale, which is much longer than the Ehrenfest time scale associated with the exponential instability of quantum motion.

[1] G. Casati and B.V. Chirikov, *Quantum Chaos* (Cambridge University Press, Cambridge, England, 1995); Physica D **86**, 220 (1995).

[2] G. Casati, B.V. Chirikov, I. Guarneri, and D.L. Shepelyansky, Phys. Rev. Lett. **56**, 2437 (1986).

[3] G. Casati, G. Maspero, and D.L. Shepelyansky, Phys. Rev. E **56**, R6233 (1997).

[4] A. Peres, Phys. Rev. A **30**, 1610 (1984).

[5] R.A. Jalabert and H.M. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001).

[6] F.M. Cucchietti, H.M. Pastawski, and D.A. Wisniacki, Phys. Rev. E. **65**, 045206(R) (2002); D.A. Wisniacki, E.G. Vergini, H. M. Pastavski, and F.M. Cucchietti, e-print nlin.CD/0111051 (unpublished).

[7] Ph. Jacquod, P.G. Silvestrov, and C.W.J. Beenakker, Phys. Rev. E **64**, 055203(R) (2001).

[8] N.R. Cerruti and S. Tomsovic, Phys. Rev. Lett. **88**, 054103 (2002).

[9] T. Prosen, Phys. Rev. E **65**, 036208 (2002).

[10] V.V. Flambaum and F.M. Izrailev, Phys. Rev. E **64**, 036220 (2001).

[11] Z.P. Karkuszewski, C. Jarzynski, and W.H. Zurek, e-print quant-ph/0111002.

[12] D.A. Wisniacki and D. Cohen, e-print quant-ph/0111125.

[13] H.M. Pastawski, P.R. Levstein, G. Usaj, J. Raya, and J. Hirschinger, Physica A **283**, 166 (2000).

[14] F. Borgonovi, G. Casati, and B. Li, Phys. Rev. Lett. **77**, 4744 (1996); F. Borgonovi, *ibid.* **80**, 4653 (1998).

[15] For $\epsilon = 0$, the numerically computed fidelity remains 1 up to times $t \approx 25$, due to roundoff errors $\epsilon \sim 10^{-14}$.

[16] G. Casati, B.V. Chirikov, I. Guarneri, and F.M. Izrailev, Phys. Lett. A **223**, 430 (1996).

[17] Strictly speaking, $f_c(t) - f_{c,\infty}$ should decay exponentially after the diffusive time scale $t_D \approx L^2/D$. However, this can hardly be seen numerically, as appears also from Fig. 5. Actually, for $t > t_D$ the fidelity is already close to its asymptotic value $f_{c,\infty} = \nu/(2\pi L) = 10^{-2}$.

[18] G. Casati, I. Guarneri, F.M. Izrailev, and R. Scharf, Phys. Rev. Lett. **64**, 5 (1990).